New symmetries for the Ablowitz-Ladik hierarchies

Da-jun Zhang,[1*] Tong-ke Ning,[2] Jin-bo Bi,[1] Deng-yuan Chen[1]
[1]*Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China*
[2]*Science College, University of Shanghai for Science and Technology, Shanghai 200093, P.R. China*

February 8, 2008

Abstract

In the letter we give new symmetries for the isospectral and non-isospectral Ablowitz-Ladik hierarchies by means of the zero curvature representations of evolution equations related to the Ablowitz-Ladik spectral problem. Lie algebras constructed by symmetries are further obtained. We also discuss the relations between the recursion operator and isospectral and non-isospectral flows. Our method can be generalized to other systems to construct symmetries for non-isospectral equations.

1 Introduction

It is well-known that infinitely many symmetries and their Lie algebra serve as one of mathematical structures of integrability for evolution equations[1]. In general, a Lax integrable isospectral evolution equation can have two sets of symmetries, isospectral and non-isospectral symmetries, or called K- and τ-symmetries, respectively. One efficient way to construct τ-symmetries was proposed by Fuchssteiner[2] by using the master symmetry. This method was later developed to many continuous (1+1)-dimensional Lax integrable systems[3, 4], (1+2)-dimensional systems[5] and further to some differential-difference cases[6, 7].

This letter will discuss K- and τ-symmetries for the isospectral Ablowitz-Ladik(AL) hierarchy, which is a well-known discrete hierarchy[8]-[11]. We will also construct new infinitely many symmetries for the non-isospectral AL hierarchy. The AL spectral problem can have two sets of isospectral hierarchies[12] which respectively correspond to positive and negative powers of the spectral parameter λ in the time-evolution part in Lax pair. The same results hold for the non-isospectral hierarchies as well, as shown in [7], where the algebraic relations between isospectral and non-isospectral flows related to positive powers of λ and the algebraic relations between isospectral and non-isospectral flows related to negative powers of λ were discussed, respectively.

Our method to construct K- and τ-symmetries for the isospectral AL hierarchy is essentially the same as used in Ref.[7, 6], and as well as a direct generalization of its continuous version[4]. Recently, we uniformed the two sets of isospectral flows (positive order and negative order) to one hierarchy with a uniform recursion operator[13]. This motivates us to do the same thing for the two sets of non-isospectral flows. Then we investigate the algebraic relations of the

*Corresponding author. E-mail: djzhang@staff.shu.edu.cn

uniformed isospectral flows and uniformed non-isospectral flows. As a result, we can generate new symmetries for those isospectral AL evolution equations and get their Lie algebra. And most important, we can construct infinitely many symmetries for the non-isospectral AL hierarchy and derive their Lie algebra. We also discuss the relations between the recursion operator and isospectral and non-isospectral flows.

The letter is organized as follows. Sec.2 lists out some basic notations. In Sec.3, we give the isospectral and non-isospectral AL hierarchies and their zero curvature representations. In Sec.4, we construct two sets of symmetries for the isospectral AL hierarchy, give their Lie algebra and discuss the relations between the recursion operator and isospectral and non-isospectral flows. In Sec.5, we construct symmetries for the non-isospectral AL hierarchy and give their Lie algebra.

2 Basic notations

To make our discussions smooth and convenient, let us redescribe some notations in [13].

Assume that $U_2 = \{u_n \equiv u(t,n) = (u_1(t,n), u_2(t,n))^T\}$ is a vector field space, where $\{u_i(t,n)\}$ are all real functions defined over $\mathbb{R} \times \mathbb{Z}$ and vanish rapidly as $|n| \to \infty$. By $\mathcal{V}_2$ denote a linear space consisting of all vector fields $f = (f_1(u(t,n)), f_2(u(t,n)))^T$ living on U_2. where $\{f_i(u(t,n))\}$ are C^∞ differentiable with respect to t and n, C^∞-Gateaux differentiable with respect to u_n, and $f_i(u(t,n))|_{u_n=0} = 0$. Then let $\mathcal{Q}_2(\lambda)$ denote a Laurent matrix polynomials space composed by all 2×2 matrixes $Q = Q(\lambda, u(t,n)) = (q_{ij}(\lambda, u(t,n)))_{2\times 2}$, where $\{q_{ij}\}$ (or Q) are all the Laurent (matrix) polynomials of λ. Besides, we define two subspaces of $\mathcal{Q}_2(\lambda)$ as

$$\mathcal{Q}_2^+(\lambda) = \{Q \in \mathcal{Q}_2(\lambda)| \text{ the lowest degree of } \lambda \geq 0\} \tag{2.1}$$

$$\mathcal{Q}_2^-(\lambda) = \{Q \in \mathcal{Q}_2(\lambda)| \text{ the highest degree of } \lambda \leq 0\}. \tag{2.2}$$

The Gateaux derivative of $f \in \mathcal{V}_2$ (or f being an operator on $\mathcal{V}_2$) in the direction $g \in \mathcal{V}_2$ is defined by

$$f'[g] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(u+\varepsilon g), \tag{2.3}$$

and the Lie product for any $f, g \in \mathcal{V}_2$ is described as

$$[\![f,g]\!] = f'[g] - g'[f]. \tag{2.4}$$

Besides, for a given discrete evolution equation $u_{nt} = K(u_n)$, $\sigma(u_n) \in \mathcal{V}_2$ is called its symmetry if

$$\sigma_t = K'[\sigma], \tag{2.5}$$

or equivalently,

$$\frac{\partial \sigma}{\partial t} = [\![K,\sigma]\!]. \tag{2.6}$$

3 Isospectral and non-isospectral AL hierarchies

The well-known AL spectral problem is given as[8]-[11]

$$E\phi = M\phi, \quad M = \begin{pmatrix} \lambda & Q_n \\ R_n & \frac{1}{\lambda} \end{pmatrix}, \quad u_n = \begin{pmatrix} Q_n \\ R_n \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3.1}$$

where E is an shift operator defined as $E^j f(n) = f(n+j)$, $\forall j \in \mathbb{Z}$. From the compatibility condition of (3.1) and its corresponding time evolution

$$\phi_t = N\phi, \;\; N = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}, \tag{3.2}$$

i.e., the zero-curvature equation

$$M_t = (EN)M - MN, \tag{3.3}$$

one can easily get[13]

$$A_n = \frac{1}{\lambda}(E-1)^{-1}(-R_n E B_n + Q_n C_n) + \frac{n\lambda_t}{\lambda} + a_0,$$

$$D_n = \lambda(E-1)^{-1}(R_n B_n - Q_n E C_n) - \frac{n\lambda_t}{\lambda} + d_0,$$

and

$$u_{nt} = (\lambda L_1 - \frac{1}{\lambda} L_2)\begin{pmatrix} B_n \\ C_n \end{pmatrix} + (a_0 - d_0)\begin{pmatrix} Q_n \\ -R_n \end{pmatrix} + \frac{(2n+1)\lambda_t}{\lambda}\begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \tag{3.4}$$

where $a_0 = A_n|_{u_n=0} - \frac{n\lambda_t}{\lambda}$, $d_0 = D_n|_{u_n=0} + \frac{n\lambda_t}{\lambda}$,

$$L_1 = \begin{pmatrix} -1 & 0 \\ 0 & E \end{pmatrix} + \begin{pmatrix} -Q_n \\ R_n E \end{pmatrix}(E-1)^{-1}(R_n, -Q_n E), \tag{3.5}$$

$$L_2 = \begin{pmatrix} -E & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -Q_n E \\ R_n \end{pmatrix}(E-1)^{-1}(R_n E, -Q_n). \tag{3.6}$$

For the iosopectral case, i.e., $\lambda_t = 0$, expanding $(B_n, C_n)^T$ in $\mathcal{Q}_2^-(\lambda)$ and $\mathcal{Q}_2^+(\lambda)$ respectively, we can get two different sets of isospectral hierarchies[12, 13]. Our method used here is little bit different from Ref.[13]. Expanding

$$\begin{pmatrix} B_n \\ C_n \end{pmatrix} = \sum_{j=0}^{l} \begin{pmatrix} b^-_{n,j} \\ c^-_{n,j} \end{pmatrix} \lambda^{-2(l-j)-1}, \qquad (l \geq 0) \tag{3.7}$$

and setting $(b^-_{n,0}, c^-_{n,0})^T \equiv (0,0)^T$ and $a_0 = -d_0 = \frac{1}{2}\lambda^{-2l}$, from (3.4) we can get

$$\begin{aligned} &u_{nt} = (1-\delta_{0,l})L_1 \begin{pmatrix} b^-_{n,l} \\ c^-_{n,l} \end{pmatrix} + \delta_{0,l}\begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \\ &L_1 \begin{pmatrix} b^-_{n,l-j} \\ c^-_{n,l-j} \end{pmatrix} = L_2 \begin{pmatrix} b^-_{n,l-j+1} \\ c^-_{n,l-j+1} \end{pmatrix}, \quad j = 1, 2, \cdots, l-1, \\ &L_2 \begin{pmatrix} b^-_{n,1} \\ c^-_{n,1} \end{pmatrix} = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}. \end{aligned}$$

Then, taking $(b^-_{n,1}, c^-_{n,1})^T = -(Q_{n-1}, R_n)^T$ yields a negative order isospectral hierarchy

$$u_{nt} = K^{(-l)} = L^{-l} K^{(0)}, \quad l = 0, 1, 2, \cdots, \tag{3.8}$$

where

$$K^{(0)} = \begin{pmatrix} Q_n \\ -R_n \end{pmatrix}, \tag{3.9}$$

and the recursion operator L is defined by

$$\begin{aligned} L = L_2 L_1^{-1} = & \begin{pmatrix} E & 0 \\ 0 & E^{-1} \end{pmatrix} + \begin{pmatrix} -Q_n E \\ R_n \end{pmatrix} (E-1)^{-1}(R_n E, Q_n E^{-1}) \\ & + \mu_n \begin{pmatrix} -EQ_n \\ R_{n-1} \end{pmatrix} (E-1)^{-1}(R_n, Q_n)\frac{1}{\mu_n}, \qquad (\mu_n = 1 - Q_n R_n). \end{aligned} \tag{3.10}$$

Similarly, expanding $(B_n, C_n)^T$ in $\mathcal{Q}_2^+(\lambda)$ as

$$\begin{pmatrix} B_n \\ C_n \end{pmatrix} = \sum_{j=0}^{l} \begin{pmatrix} b_{n,j}^+ \\ c_{n,j}^+ \end{pmatrix} \lambda^{2(l-j)+1}, \qquad (l \geq 0), \tag{3.11}$$

setting $(b_{n,0}^+, c_{n,0}^+)^T \equiv (0,0)^T$ and taking $a_0 = -d_0 = \frac{1}{2}\lambda^{2l}$ and $(b_{n,1}^+, c_{n,1}^+)^T = (Q_n, R_{n-1})^T$, we can get another isospectral hierarchy, i.e., a positive order hierarchy,

$$u_{nt} = K^{(l)} = L^l K^{(0)}, \quad l = 0, 1, 2, \cdots . \tag{3.12}$$

Thus, (3.8) and (3.12) can be uniformed to one hierarchy as[13]

$$u_{nt} = K^{(l)} = L^l K^{(0)}, \quad l \in \mathbb{Z}. \tag{3.13}$$

The recursion operator L is a strong and hereditary symmetry operator for the above hierarchy[13], and this hierarchy has been shown to have multi-Hamiltonian structures[12, 13] and infinitely many conservation laws[14].

For the non-isospectral case, i.e., $\lambda_t \neq 0$, we first expanding $(B_n, C_n)^T$ in $\mathcal{Q}_2^-(\lambda)$ as (3.7), setting $(b_{n,0}^-, c_{n,0}^-)^T \equiv (0,0)^T$ and taking $a_0 = -d_0 = 0$, $\lambda_t = \lambda^{-2l+1}$ and $(b_{n,1}^-, c_{n,1}^-)^T = -((2n-1)Q_{n-1}, (2n+1)R_n)^T$, we can get the negative order non-isospectral hierarchy

$$u_{nt} = \sigma^{(-l)} = L^{-l}\sigma^{(0)}, \quad l = 0, 1, 2, \cdots , \tag{3.14}$$

where

$$\sigma^{(0)} = (2n+1)\begin{pmatrix} Q_n \\ -R_n \end{pmatrix}. \tag{3.15}$$

Expanding $(B_n, C_n)^T$ in $\mathcal{Q}_2^+(\lambda)$ as (3.11), setting $(b_{n,0}^+, c_{n,0}^+)^T \equiv (0,0)^T$ and taking $a_0 = -d_0 = 0$, $\lambda_t = \lambda^{2l+1}$, and $(b_{n,1}^+, c_{n,1}^+)^T = ((2n+1)Q_n, (2n-1)R_{n-1})^T$, we get the positive order non-isospectral hierarchy

$$u_{nt} = \sigma^{(l)} = L^l \sigma^{(0)}, \quad l = 0, 1, 2, \cdots , \tag{3.16}$$

and these two non-isospectral hierarchies can be uniformed to one hierarchy as

$$u_{nt} = \sigma^{(l)} = L^l \sigma^{(0)}, \quad l \in \mathbb{Z}. \tag{3.17}$$

$\{K^{(l)}\}$ and $\{\sigma^{(l)}\}$ are called isospectral flows and non-isospectral flows, respectively. On the basis of the above discussions, it is not difficult to give the zero curvature representations of these two sets of flows.

Proposition 3.1 *$\{K^{(l)}\}$ and $\{\sigma^{(l)}\}$ have the following zero curvature representations*

$$M'[K^{(l)}] = (EN^{(l)})M - MN^{(l)}, \tag{3.18}$$

$$M'[\sigma^{(l)}] = (EU^{(l)})M - MU^{(l)} - M_\lambda \lambda_t, \tag{3.19}$$

where

$$\lambda_t = \lambda^{2l+1}, \quad (l \in \mathbb{Z}), \tag{3.20}$$

$K^{(l)}, \sigma^{(l)} \in \mathcal{V}_2$, $N^{(l)}, U^{(l)} \in \mathcal{Q}_2(\lambda)$ and satisfy

$$N^{(l)}|_{u_n=0} = \frac{1}{2}\lambda^{2l}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U^{(l)}|_{u_n=0} = \lambda^{2l}\begin{pmatrix} n & 0 \\ 0 & -n \end{pmatrix}, \quad (l \in \mathbb{Z}). \tag{3.21}$$

□

Besides, we have the following properties[13] on the AL spectral problem (3.1).

Proposition 3.2 *The following matrix equation*

$$M'[X] = (EN)M - MN, \quad X \in \mathcal{V}_2,\ N \in \mathcal{Q}_2(\lambda) \text{ and } N|_{u_n=0} = 0 \tag{3.22}$$

has only zero solutions $X = 0$ and $N = 0$. □

Proposition 3.3 *For any given $Y \neq 0 \in \mathcal{V}_2$, there exist solutions $N^\pm \in \mathcal{Q}_2^\pm(\lambda)$ satisfying*

$$M'[L^{\pm1}Y - \lambda^{\pm2}Y] = (EN^\pm)M - MN^\pm, \quad N^\pm|_{u_n=0} = 0 \tag{3.23}$$

where L^{+1} denotes L defined by (3.10). □

4 Symmetries for the isospectral AL hierarchy and Lie algebra

In this section, we discuss the algebras of the flows $\{K^{(l)}\}$ and $\{\sigma^{(l)}\}$ and the related symmetries for the isospectral AL hierarchy. We also give the relations between the recursion operator L and these two sets of flows.

4.1 Symmetries for the isospectral AL hierarchy and Lie algebra

In this subsection, we will directly generate the method used in Ref.[4] to construct K- and τ-symmetries for the isospectral AL hierarchy. Our method is also essentially the same as used in Ref.[6, 7].

We list out our results through the following propositions.

Proposition 4.1 *If the isospectral and non-isospectral flows $\{K^{(l)}\}$ and $\{\sigma^{(l)}\}$ have their zero curvature representations* (3.18) *and* (3.19)*, then, $\forall l, s \in \mathbb{Z}$, the Lie products of these flows satisfy*

$$\begin{aligned}
&M'[[\![K^{(l)}, K^{(s)}]\!]] = (E < N^{(l)}, N^{(s)} >)M - M < N^{(l)}, N^{(s)} >, \\
&M'[[\![K^{(l)}, \sigma^{(s)}]\!]] = (E < N^{(l)}, U^{(s)} >)M - M < N^{(l)}, U^{(s)} >, \\
&M'[[\![\sigma^{(l)}, \sigma^{(s)}]\!]] = (E < U^{(l)}, U^{(s)} >)M - M < U^{(l)}, U^{(s)} > -2(s-l)M_\lambda \lambda^{2(l+s)+1},
\end{aligned} \tag{4.1}$$

where

$$
\begin{aligned}
&< N^{(l)}, N^{(s)} >= N^{(l)\prime}[K^{(s)}] - N^{(s)\prime}[K^{(l)}] + [N^{(l)}, N^{(s)}],\\
&< N^{(l)}, U^{(s)} >= N^{(l)\prime}[\sigma^{(s)}] - U^{(s)\prime}[K^{(l)}] + [N^{(l)}, U^{(s)}] + N_\lambda^{(l)}\lambda^{2s+1},\\
&< U^{(l)}, U^{(s)} >= U^{(l)\prime}[\sigma^{(s)}] - U^{(s)\prime}[\sigma^{(l)}] + [U^{(l)}, U^{(s)}] + U_\lambda^{(l)}\lambda^{2s+1} - U_\lambda^{(s)}\lambda^{2l+1},
\end{aligned} \tag{4.2}
$$

and satisfy

$$
\begin{aligned}
&< N^{(l)}, N^{(s)} > |_{u_n=0} = 0,\\
&< N^{(l)}, U^{(s)} > |_{u_n=0} = 2lN^{(l+s)}|_{u_n=0},\\
&< U^{(l)}, U^{(s)} > |_{u_n=0} = 2(l-s)U^{(l+s)}|_{u_n=0}.
\end{aligned} \tag{4.3}
$$

Here, $[A, B] = AB - BA$.

Proof: A similar proof procedure can be found in Ref.[4]. (4.1) and (4.2) can be derived from the zero curvature representations (3.18) and (3.19) by making use of the identity

$$
M'[[\![f, g]\!]] = (M'[f])'[g] - (M'[g])'[f], \quad \forall f, g \in \mathcal{V}_2. \tag{4.4}
$$

(4.3) can be obtained by using the asymptotic conditions (3.21). □

Then, using Proposition 3.1 and 3.2, we can have the following result.

Proposition 4.2 *The isospectral and non-isospectral flows* $\{K^{(l)}\}$ *and* $\{\sigma^{(l)}\}$ *form a Lie algebra* $\mathcal{F}$ *through the Lie product* $[\![\cdot, \cdot]\!]$*; and* $\forall l, s \in \mathbb{Z}$ *they have the following relations*

$$
\begin{aligned}
&[\![K^{(l)}, K^{(s)}]\!] = 0,\\
&[\![K^{(l)}, \sigma^{(s)}]\!] = 2lK^{(l+s)},\\
&[\![\sigma^{(l)}, \sigma^{(s)}]\!] = 2(l-s)\sigma^{(l+s)}.
\end{aligned} \tag{4.5}
$$

□

Different from Ref.[7], here we have expanded the Lie product relations between $\{K^{(l)}\}$ and $\{\sigma^{(s)}\}$ for any integer subindices. This can result in some interesting relations. For example, $\forall s \in \mathbb{Z}$, we have

$$
\begin{aligned}
&[\![K^{(0)}, \sigma^{(l)}]\!] \equiv 0, \quad [\![K^{(l)}, \sigma^{(0)}]\!] = 2lK^{(l)},\\
&[\![K^{(l)}, \sigma^{(-l)}]\!] = 2lK^{(0)}, \quad [\![\sigma^{(l)}, \sigma^{(0)}]\!] = 2l\sigma^{(l)}, \quad [\![\sigma^{(l)}, \sigma^{(-l)}]\!] = 4l\sigma^{(0)}.
\end{aligned}
$$

Besides, by virtue of the above proposition, for any equation in the isospectral hierarchy (3.13), it is not difficult to get two sets of symmetries and their Lie algebra.

Proposition 4.3 *The arbitrary member in the isospectral hierarchy* (3.13)*,*

$$
u_{nt} = K^{(l)}, \quad \forall l \in \mathbb{Z}, \tag{4.6}
$$

has the following two sets of symmetries,

$$
\{K^{(s)}\} \ \text{ and } \ \{\tau^{(l,s)} = 2ltK^{(l+s)} + \sigma^{(s)}\}, \qquad s \in \mathbb{Z}, \tag{4.7}
$$

which we call K*-symmetries and* τ*-symmetries, respectively. They form a Lie algebra* $\mathcal{S}$ *and have the relations*

$$
\begin{aligned}
&[\![K^{(q)}, K^{(s)}]\!] = 0,\\
&[\![K^{(q)}, \tau^{(l,s)}]\!] = 2qK^{(q+s)},\\
&[\![\tau^{(l,q)}, \tau^{(l,s)}]\!] = 2(l-s)\tau^{(l,q+s)}.
\end{aligned} \tag{4.8}
$$

$\square$

From the Lie product relations (4.5) and (4.8), it is not difficult to find the generators of the Lie algebras $\mathcal{F}$ and $\mathcal{S}$.

Proposition 4.4 *The Lie algebra $\mathcal{F}$ can be generated by the following four generators,*

$$\sigma^{(2)}, \quad \sigma^{(-2)}, \quad \sigma^{(1)}(\text{or } \sigma^{(-1)}), \quad K^{(1)}(\text{or } K^{(-1)}).$$

The Lie algebra $\mathcal{S}$ can be generated by the following four generators,

$$\tau^{(l,2)}, \quad \tau^{(l,-2)}, \quad \tau^{(l,1)}(\text{or } \tau^{(l,-1)}), \quad K^{(1)}(\text{or } K^{(-1)}).$$

$\square$

4.2 Relations between the recursion operator and flows

In this subsection, we discuss the relations between the recursion operator and flows.

Proposition 4.5 *For any $l \in \mathbb{Z}$, the flows $K^{(l)}$ and $\sigma^{(l)}$ and their recursion operator L satisfy*

$$L'[K^{(l)}] - [K^{(l)\prime}, L] = 0, \tag{4.9}$$

$$L'[\sigma^{(l)}] - [\sigma^{(l)\prime}, L] - 2L^{l+1} = 0. \tag{4.10}$$

Proof: (4.9) has been proved in Ref.[13]. For (4.10), we prove

$$(L'[\sigma^{(l)}] - [\sigma^{(l)\prime}, L] - 2L^{l+1})Y = 0, \quad \forall Y \in \mathcal{V}_2, \quad \forall l \in \mathbb{Z}, \tag{4.11}$$

i.e.,

$$L[\![\sigma^{(l)}, Y]\!] - [\![\sigma^{(l)}, LY]\!] - 2L^{l+1}Y = 0, \quad \forall Y \in \mathcal{V}_2, \quad \forall l \in \mathbb{Z}. \tag{4.12}$$

$\forall Y \in \mathcal{V}_2$, in the light of Proposition 3.3, there exists N^+ and $W^{(l)}$ in $\mathcal{Q}_2(\lambda)$ such that

$$M'[LY - \lambda^2 Y] = (EN^+)M - MN^+, \quad N^+|_{u_n=0} = 0, \tag{4.13}$$

$$M'[L[\![\sigma^{(l)}, Y]\!]] = (EW^{(l)})M - MW^{(l)} + \lambda^2 M'[[\![\sigma^{(l)}, Y]\!]], \quad W^{(l)}|_{u_n=0} = 0. \tag{4.14}$$

Meanwhile, using zero curvature representation of $\sigma^{(l)}$, i.e., (3.19), and the identity (4.4), we have

$$\begin{aligned} M'[[\![\sigma^{(l)}, Y]\!]] = \; & (EU^{(l)\prime}[Y])M + (EU^{(l)})M'[Y] - M'[Y]U^{(l)} - MU^{(l)\prime}[Y] \\ & - \lambda^{2l+1}M'_\lambda[Y] - (M'[Y])'[\sigma^{(l)}]. \end{aligned} \tag{4.15}$$

On the other hand, using the identity (4.4), (4.13) and zero curvature representation (3.19), we can have

$$\begin{aligned} M'[[\![\sigma^{(l)}, LY]\!]] = \; & \Big[E(U^{(l)\prime}[LY] - N^{+\prime}[\sigma^{(l)}] + [U^{(l)}, N^+] - \lambda^{2l+1}N^+_\lambda)\Big]M \\ & - M\Big(U^{(l)\prime}[LY] - N^{+\prime}[\sigma^{(l)}] + [U^{(l)}, N^+] - \lambda^{2l+1}N^+_\lambda\Big) \\ & + \lambda^2\Big[(EU^{(l)})M'[Y] - M'[Y]U^{(l)} - \lambda^{2l}M'[Y] - \lambda^{2l+1}M'_\lambda[Y] - (M'[Y])'[\sigma^{(l)}]\Big]. \end{aligned} \tag{4.16}$$

Then (4.16) and (4.14) together with (4.15) yield

$$M'[L[\![\sigma^{(l)}, Y]\!] - [\![\sigma^{(l)}, LY]\!] - 2\lambda^{2(l+1)}Y] = (E < W^{(l)}, U^{(l)}, N^{+} >)M - M < W^{(l)}, U^{(l)}, N^{+} >, \tag{4.17}$$

where

$$< W^{(l)}, U^{(l)}, N^{+} >= W^{(l)} - U^{(l)\prime}[LY - \lambda^{2}Y] + N^{+\prime}[\sigma^{(l)}] + [N^{+}, U^{(l)}] + \lambda^{2l+1}N^{+}_{\lambda} \tag{4.18}$$

satisfying

$$< W^{(l)}, U^{(l)}, N^{+} > |_{u_n=0} = 0. \tag{4.19}$$

Then, noting that (4.13) implies that there exists $\widetilde{N}^{+} \in \mathcal{Q}_2(\lambda)$ such that

$$M'[L^{l+1}Y - \lambda^{2(l+1)}Y] = (E\widetilde{N}^{+})M - M\widetilde{N}^{+}, \quad \widetilde{N}^{+}|_{u_n=0} = 0, \tag{4.20}$$

and using Proposition 3.2, we can finally reach the equality (4.12) and thus we complete the proof. □

We note that the algebra relations (4.5) can also be obtained through the reductive approach by using (4.9) and (4.10).

5 Symmetries for the non-isospectral AL hierarchy and Lie algebra

By virtue of the Lie product relations given in Proposition 4.2, we can construct infinitely many symmetries for any member in the non-isospectral AL hierarchy (3.17).

Proposition 5.1 *For any $l \in \mathbb{Z}$, the non-isospectral evolution equation*

$$u_{nt} = \sigma^{(l)} \tag{5.1}$$

has the following infinitely many symmetries

$$\eta^{(l,m)} = \sum_{j=0}^{m} C_m^j (2lt)^{m-j} \sigma^{(l-jl)}, \quad (m = 0, 1, 2, \cdots), \tag{5.2}$$

$$\gamma^{(l,m)} = \sum_{j=0}^{m} C_m^j (2lt)^{m-j} K^{(-jl)}, \quad (m = 0, 1, 2, \cdots), \tag{5.3}$$

where $C_m^j = \frac{m!}{j!(m-j)!}$. For convenient, we call (5.2) and (5.3) the η-symmetries and γ-symmetries, respectively. □

This proposition can be proved by direct verification according to the definition (2.6) and the algebraic relations (4.5).

Proposition 5.2 *η-symmetries $\{\eta^{(l,m)}\}_{m=0,1,2}$ and γ-symmetries $\{\gamma^{(l,m)}\}_{m=0,1,2,\cdots}$ construct a Lie algebra $\widetilde{\mathcal{S}}$ and they follow the following Lie product relations,*

$$\begin{array}{ll}
[\![\eta^{(l,m)}, \eta^{(l,m)}]\!] = 0, & (m = 0, 1, 2), \\
[\![\eta^{(l,m)}, \eta^{(l,s)}]\!] = 2(s-m)l\eta^{(l,s+m-1)}, & (m, s = 0, 1, 2, \cdots, \ m \neq s), \\
[\![\gamma^{(l,m)}, \gamma^{(l,s)}]\!] = 0, & (m, s = 0, 1, 2, \cdots), \\
[\![\eta^{(l,m)}, \gamma^{(l,0)}]\!] = 0, & (m = 0, 1, 2, \cdots), \\
[\![\eta^{(l,m)}, \gamma^{(l,s)}]\!] = 2sl\gamma^{(l,s+m-1)}, & (m = 0, 1, 2, \cdots, \ s = 1, 2, \cdots).
\end{array} \tag{5.4}$$

Obviously, the Lie algebra $\widetilde{\mathcal{S}}$ has three generators

$$\eta^{(l,0)},\ \eta^{(l,3)},\ \gamma^{(l,1)}. \tag{5.5}$$

Proof: We only prove the second and the last equalities in (5.4). From (5.2) we have

$$\begin{aligned}
[\![\eta^{(l,m)},\eta^{(l,s)}]\!] &= \sum_{j=0}^{m}\sum_{h=0}^{s} C_m^j C_s^h (2lt)^{m+s-j-h}[\![\sigma^{(l-jl)},\sigma^{(l-hl)}]\!]\\
&= 2l\sum_{j=0}^{m}\sum_{h=0}^{s} C_m^j C_s^h (2lt)^{m+s-j-h}(h-j)\sigma^{(l-(j+h-1)l)}\\
&= 2l\sum_{i=1}^{m+s}\sum_{j=0}^{min\{i,m\}} C_m^j C_s^{i-h}(2lt)^{m+s-i}(i-2j)\sigma^{(l-(i-1)l)}.
\end{aligned}$$

Without loss of generality, we let $m>s$. Then, by noting that

$$\sum_{j=0}^{min\{i,m\}} C_m^j C_s^{i-h}(2lt)^{m+s-i}(i-2j) = (s-m)C_{m+s-1}^{i-1},\quad (m,s=0,1,\cdots,\quad m>s,\quad 1\le i\le m+s), \tag{5.6}$$

we immediately get

$$[\![\eta^{(l,m)},\eta^{(l,s)}]\!] = 2(s-m)l\sum_{i=0}^{m+s-1} C_{m+s-1}^{i}\sigma^{(l-il)} = 2(s-m)l\eta^{(l,s+m-1)}.$$

Similarly, we can prove the last equality in (5.4), where we need to use the identity

$$\sum_{j=0}^{min\{i,m\}} C_m^j C_s^{i-h}(2lt)^{m+s-i}(i-j) = sC_{m+s-1}^{i-1},\quad (m,s=0,1,\cdots,\quad m>s,\quad 1\le i\le m+s). \tag{5.7}$$

The proof for (5.6) and (5.7) will be given in Appendix. □

In addition, for the isospectral equation $u_{nt}=K^{(-l)}$ and non-isospectral equation $u_{nt}=\sigma^{(l)}$, they have a non-trivial mutual symmetry,

$$\sigma = -2ltK^{(0)} - K^{(-l)} + \sigma^{(l)}. \tag{5.8}$$

Conclusion

To sum up, in this letter, we first respectively uniformed the isospectral AL hierarchy and non-isospectral AL hierarchy. Then we derived Lie algebraic relations of these uniformed flows by means of their zero curvature representations, and consequently we obtained K-symmetries and τ-symmetries for the isospectral AL hierarchy. As the Lie product relations between $\{K^{(l)}\}$ and $\{\sigma^{(s)}\}$ have been expended for any integer subindices l and s, some obtained symmetries are new. And, as an important result, we worked out new infinitely many symmetries for the non-isospectral AL hierarchy and gave their Lie algebra. Generators of these obtained Lie algebras have been given. The relations between the recursion operator L and the two sets of flows

were also discussed. It is known that it is not easy to construct infinitely many symmetries for non-isospectral evolution equations. We believe that our method to derive symmetries for non-isospectral equations through constructing negative order hierarchies is general and can apply to other systems. This will be investigated in detail elsewhere. In fact, there have been some known systems with inverse recursion operators, i.e., with positive and negative order hierarchies[15]-[18]. In addition, can our new symmetries lead to new reductions and solutions?

Acknowledgments

This project is supported by the National Natural Science Foundation of China (10371070), the Youth Foundation of Shanghai Education Committee and the Foundation of Shanghai Education Committee for Shanghai Prospective Excellent Young Teachers.

References

[1] A.S. Fokas, Symmetries and integeability, Stud. Appl. Math., **77** (1987) 253-299.

[2] B Fuchssteiner, Master symmetries, higher order time-dependent symmetries and conserved densities of nonlinear evolution equations, Prog. Theo. Phys., **70** (1983) 1508-1522.

[3] D.Y. Chen, H.W. Zhang, Lie algebraic structure for the AKNS system, J. Phys. A: Gen. Math. Phys., **24** (1991) 377-383.

[4] D.Y. Chen, D.J. Zhang, Lie algebraic structures of (1+1)-dimensional Lax integrable systems, J. Math. Phys., **37** (1996) 5524-5538.

[5] D.Y. Chen, H.W. Xin, D.J. Zhang, Lie algebraic structures of some (1+2)-dimensional Lax integrable systems, Chaos, Solitons and Fractals, **15** (2003) 761-770.

[6] W. X. Ma, B. Fuchssteiner, Algebraic structure of discrete zero curvature equations and master symmetries of discrete evolution equations, J. Math. Phys., **40** (1999) 2400-2418.

[7] K.M. Tamizhmani, W.X. Ma, Master symmetries from Lax operators for certain lattice soliton hierarchies, J. Phys. Soc. Jpn., **69** (2000) 351-361.

[8] M.J. Ablowitz and J.F. Ladik, Nonlinear differential-difference equations, J. Math. Phys., **16** (1975) 598-603.

[9] M.J. Ablowitz and J.F. Ladik, Nonlinear differential-difference equations and Fourier analysis, J. Math. Phys., **17** (1976) 1011-1018.

[10] M.J. Ablowitz and J.F. Ladik, A nonlinear difference scheme and inverse scattering, Stud. Appl. Math., **55** (1976) 213-229.

[11] M.J. Ablowitz and J.F. Ladik, On the solution of a class of nonlinear partial difference equation, Stud. Appl. Math., **57** (1977) 1-12.

[12] Y.B. Zeng, S.R. Wojciechowski, Restricted flows of the Ablowitz-Ladik hierarchy and their continuous limits, J. Phys. A: Math. Gen., **28** (1995) 113-134.

[13] D.J. Zhang, D.Y. Chen, Hamiltonian structure of discrete soliton systems, J. Phys. A: Math. Gen., **35** (2002) 7225-7241.

[14] D.J. Zhang, D.Y. Chen, The conservation laws of some discrete soliton systems, Chaos, Solitons and Fractals, **14**, (2002) 573-579.

[15] Z.J. Qiao, W. Strampp, Negative order MKdV hierarchy and a new integrable Neumann-like system, Physica A, **313** (2002) 365-380.

[16] R.G. Zhou, Hierarchy of negative order equation and its Lax pair, J. Math. Phys., **36** (1995) 4220-4225.

[17] W.X. Ma, R.G. Zhou, On inverse recursion operator and tri-Hamiltonian formulation for a Kaup-Newell system of DNLS equations, J. Phys. A: Math. Gen., **32** (1999) L239-L242.

[18] Y.P. Sun, D.Y. Chen, X.X. Xu, Positive and negative hierarchies of nonlinear integrable lattice models and three integrable coupling systems associated with a discrete spectral problem, accepted and to be published in Nonlinear Analysis, 2006.

Appendix: Proof for (5.6) **and** (5.7)

First, expanding both sides of the identity

$$(1+x)^m(1+x)^s = (1+x)^{m+s}$$

and picking up the coefficients of x^i on both sides, we can easily get

$$\sum_{j=0}^{min\{m,i\}} C_m^j C_s^{i-j} = C_{m+s}^i, \quad (m, s = 0, 1, 2, \cdots, \quad 0 \le i \le m+s). \tag{A.1}$$

Similarly, expanding both sides of the identity

$$\frac{d(1+x)^m}{dx}(1+x)^s = m(1+x)^{m+s-1}$$

and picking up the coefficients of x^{i-1} on both sides yield

$$\sum_{j=0}^{min\{m,i\}} jC_m^j C_s^{i-j} = mC_{m+s-1}^{i-1}, \quad (s = 0, 1, \cdots, \quad m = 1, 2, \cdots, \quad 1 \le i \le m+s). \tag{A.2}$$

Next, by noting that $iC_{m+s}^i = (m+s)C_{m+s-1}^{i-1}$, from $(A.1)$ we have

$$\sum_{j=0}^{min\{m,i\}} iC_m^j C_s^{i-j} = (m+s)C_{m+s-1}^{i-1}, \quad (s = 0, 1, \cdots, \quad m = 1, 2, \cdots, \quad 1 \le i \le m+s). \tag{A.3}$$

It then follows from $(A.2)$ and $(A.3)$ that

$$\sum_{j=0}^{min\{m,i\}} (i-2j)C_m^j C_s^{i-j} = (s-m)C_{m+s-1}^{i-1}, \quad (s = 0, 1, \cdots, \quad m = 1, 2, \cdots, \quad 1 \le i \le m+s) \tag{A.4}$$

and

$$\sum_{j=0}^{min\{m,i\}} (i-j)C_m^j C_s^{i-j} = sC_{m+s-1}^{i-1}, \quad (s = 0, 1, \cdots, \quad m = 1, 2, \cdots, \quad 1 \le i \le m+s), \tag{A.5}$$

i.e., (5.6) and (5.7).